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Basket Options on Heterogeneous Underlying Assets

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Abstract. Basket options are among the most popular products of the new generation of exotic options. This attraction is explained by the fact that they can efficiently and simultaneously hedge a wide variety of intrinsically different financial risks. They are flexible enough to include all the risks faced by non-financial arms. Unfortunately, the existing literature on basket options considers only homogeneous baskets where all the underlying assets are identical and hedge the same kind of risk. Moreover, the empirical implementation of basket-option models is not yet well developed, particularly when they are composed of heterogeneous underlying assets. This paper focus on the modelization and the parameters estimation of basket options on commodity price with stochastic convenience yield, exchange rate, and domestic and foreign zero-coupon bonds in a stochastic interest rates setting. We empirically compare the performance of the heterogeneous basket option to that of a portfolio of individual options. The results show that the basket strategy is less expensive and more efficient. We apply the maximum-likelihood method to estimate the different parameters of the theoretical basket model as well as the correlations between the variables. Monte Carlo studies are conducted to examine the performance of the maximum-likelihood estimator in finite samples of simulated data. A real data study is presented.

Keywords. Basket options, maximum likelihood, hedging performance, options pricing, Monte Carlo simulation.

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1 Introduction

The vast majority of non financial firms faces different financial risks (interest rates, exchange rates, commodity prices, etc.) and would like more efficient and cheaper ways to hedge. Traditionally, these firms use derivative securities to hedge each of these risks separately. A portfolio approach (like the basket option) allows the inclusion of correlations between these risks. Usually traded over the counter, the design of the basket option is made to meet the specific needs of the firm and, when the underlying basket is well diversified, its theoretical price is lower than the price of a basket of individual options. However, in practice, it may be difficult to find a counterpart (usually a bank) and the latter requires high premiums for these options due to the lack of liquidity.

The existing literature on basket options (Gentle (1993), Curran (1994), Huynh (1994), Barraquand (1995), Milevsky and Posner (1998a, 1998b), Posner and Milevsky (1999), Dahl (2000), Dahl and Benth (2001), Flamouris and Giamouridis (2007), Pellizzari (2001), Wan (2002), Ju (2002), Datey *et al.* (2003), Deelstra *et al.* (2004), Vanmaele *et al.* (2004), Brigo *et al.* (2004), and Laurence and Wang (2004)) is proposing different ways to price an homogenous basket, that is, the underlying portfolio is constituted of identical assets (such as exchange rates or equities) usually modeled with multidimensional geometrical Brownian motions with constant spot interest rate. Unfortunately, this type of basket does not always correspond to a firm's needs.

Going beyond the existing recent papers, the focus of our work will be on modeling, performance analysis, and estimation of parameters related to basket options on heterogeneous underlying assets. Our main contribution consists in considering a basket option on multiple underlying assets which are intrinsically different. In the same basket, we combine commodity prices, exchange rates, and zero-coupon bonds. The basket option we propose allows

non-financial firms to cover some of their financial exposure with a single hedge and at a lower cost than if the company were to hedge each of these risks separately. This paper treats all the aspects related to basket options, such as modelization and empirical implementation of the theoretical model, making our contributions very useful, especially for practitioners who use this kind of product for hedging. To our knowledge, it is the first time that an heterogeneous basket composed with intrinsically different assets including stochastic interest rates is considered. Moreover, in this extended framework, the estimation of the model's parameters is non trivial and is required for a practical use of these options.

The first objective of this paper consists in developing a theoretical model for a basket option under the equivalent martingale measure. As to be later justified, we suppose that the commodity price and the convenience yield share the same source of risk, which allows us to work with a complete market and adopt a single price for the basket option. This simplification frees us from having to define and estimate a functional form for the market-price risk associated with the stochastic convenience yield.

Second, we compare the performance of a basket option to that of a portfolio of individual plain vanilla options by computing option prices and profits. We prove empirically that the heterogeneous basket option costs less and is more efficient. Given that our model depends on several underlying assets with different stochastic processes we do not obtain a closed-form solution for the price of the basket option. Hence, we carry out a Monte Carlo simulation to price the basket option.

Concerning the empirical implementation of the basket-option model, one of the main difficulties is that some variables, such as the convenience yield and the instantaneous forward rates, are not directly observable. A well-suited technique to deal with such situations is to use the maximum-likelihood method. The main advantage of using the maximum-

likelihood approach to estimate basket parameters comes from the asymptotic properties of its estimator, properties such as consistency and normality. These properties are necessary for statistical inference, because they make it possible to build confidence intervals when applying maximum likelihood to real data. In this paper, we use this technique to estimate all the parameters of the basket model as well as the correlations between the underlying assets composing the basket. This estimation procedure is implemented empirically on simulated data, and its performance is analyzed using a Monte Carlo study. We also use real data on commodity prices, exchange rates, and futures on zero-coupon bonds to estimate the different parameters of the basket model.

The remainder of the paper is organized as follows. Section 2 presents the model including the commodity with stochastic convenience yield, exchange rate and stochastic domestic and foreign interest rates chosen among the Heath, Jarrow and Morton (hereafter HJM) family. In Section 3, the performances of the basket option and a portfolio of individual options are compared numerically. The basket option is priced using Monte Carlo simulation. Section 4 discusses the parameters' estimation using the maximum likelihood framework. A Monte Carlo study analyses the performance of the estimators. A study using real data is also presented. Section 5 concludes.

2 The model

Let S_t denotes the commodity price at time t expressed in the domestic currency and δ_t represents its stochastic convenience yield.[1] This model is inspired from Schwartz (1997), at the difference that both processes share the same source of risk. Indeed, allowing for

[1] The convenience yield of a given commodity is defined as the flow of services that accrues to a holder of the physical commodity, but not to a holder of a contract for future delivery of the same commodity (Brennan 1991).

stochastic convenience yield with an extra source of noise will leads to an incomplete model, since the convenience yield is not a tradable asset. Our simplification solves this problem and may be justified with a highly positive correlation between the commodity return and its convenience yield (see Brennan (1991)). The exchange rate C_t is the value at time t of one unit of the foreign currency expressed in the domestic currency. The instantaneous forward rates' models ($f(t,T)$ denotes the domestic rate and $f^*(t,T)$ stands for the foreign rate) are chosen among the HJM family where the volatility parameters $\eta_{t,T}$ and $\eta^*_{t,T}$ are deterministic functions[2] of time t and maturity T. Under the objective measure P, the model is

$$dS_t = S_t\left[(\alpha_s - \delta_t)\,dt + \sigma_s dW_t^{(1)}\right], \tag{1a}$$

$$d\delta_t = \kappa(\theta - \delta_t)dt + \sigma_\delta dW_t^{(1)}, \tag{1b}$$

$$dC_t = C_t\left[\alpha_c dt + \sigma_c dW_t^{(2)}\right], \tag{1c}$$

$$df(t,T) = \gamma_{t,T}dt + \eta_{t,T}dW_t^{(3)}, \tag{1d}$$

$$df^*(t,T) = \gamma^*_{t,T}dt + \eta^*_{t,T}dW_t^{(4)}, \tag{1e}$$

where $\left\{\mathbf{W}_t = \left(W_t^{(1)}, W_t^{(2)}, W_t^{(3)}, W_t^{(4)}\right) : t \geq 0\right\}$ is a four dimensional $P-$Brownian motion with a constant correlation matrix $\boldsymbol{\varrho} = \left(\rho_{ij}\right)_{i,j\in\{1,2,3,4\}}$. The parameters $\alpha_s, \sigma_s, \kappa, \theta, \sigma_\delta, \alpha_c, \sigma_c$ are unknown and need to be estimated. The deterministic functions $\gamma_{t,T}, \gamma^*_{t,T}, \eta_{t,T}, \eta^*_{t,T}$ will be specified and estimated as well in Section 4. Note that both instantaneous forward rates are Gaussian processes allowing for potential negative interest rates.

We consider some zero coupon bonds paying one unit of their currency at time T. According to Equations (1d) and (1e), the time t values of the domestic and foreign zero coupon

[2] Although it is possible to develop the pricing model in this general setting, the functions $\eta_{t,T}$ and $\eta^*_{t,T}$ will be set to some constants (η and η^*) or some exponential functions ($\eta\exp(\tau t)$) and $\eta^*\exp(\tau^* t)$) at the estimation stage.

bonds follow respectively

$$dP(t,T) = P(t,T)\left[\left(r_t - \beta_{t,T}\left(\frac{\gamma_{t,T}}{\eta_{t,T}} - \beta_{t,T}\right)\right)dt - \beta_{t,T}dW_t^{(3)}\right], \tag{1f}$$

$$dP^*(t,T) = P^*(t,T)\left[\left(r_t^* - \beta_{t,T}^*\left(\frac{\gamma_{t,T}^*}{\eta_{t,T}^*} - \beta_{t,T}^*\right)\right)dt - \beta_{t,T}^* dW_t^{(4)}\right] \tag{1g}$$

where $r_t = f(t,t)$ and $r_t^* = f^*(t,t)$ are respectively the domestic and the foreign spot interest rates at time t and $\beta_{t,T} = \int_t^T \eta_{t,s}ds$, $\beta_{t,T}^* = \int_t^T \eta_{t,s}^* ds$. Finally, the time t value of the domestic and foreign bank accounts are characterized respectively by $dD_t = r_t D_t dt$ and $dD_t^* = r_t^* D_t^* dt$.

Following the classical approach of risk neutral evaluation[3], the model is obtained under the unique risk neutral measure Q :

$$dS_t = S_t\left[(r_t - \delta_t)\,dt + \sigma_s d\widetilde{W}_t^{(1)}\right] \tag{2a}$$

$$d\delta_t = (\kappa\theta - \frac{\sigma_\delta}{\sigma_s}(\alpha_s - r_t) - \kappa\delta_t)dt + \sigma_\delta d\widetilde{W}_t^{(1)} \tag{2b}$$

$$dC_t = C_t\left[(r_t - r_t^*)\,dt + \sigma_c d\widetilde{W}_t^{(2)}\right] \tag{2c}$$

$$dP(t,T) = P(t,T)\left[r_t dt - \beta_{t,T}d\widetilde{W}_t^{(3)}\right] \tag{2d}$$

$$dP^*(t,T) = P^*(t,T)\left[\left(r_t^* + \beta_{t,T}^*\sigma_c\rho_{2,4}\right)dt - \beta_{t,T}^* d\widetilde{W}_t^{(4)}\right] \tag{2e}$$

where $\left\{\widetilde{\mathbf{W}}_t = \left(\widetilde{W}_t^{(1)}, \widetilde{W}_t^{(2)}, \widetilde{W}_t^{(3)}, \widetilde{W}_t^{(4)}\right) : t \geq 0\right\}$ is a four dimensional $Q-$Brownian motion with a constant correlation matrix $\boldsymbol{\varrho}$.

[3] In order to determine the risk free measure Q, one need to constitute the self financing assets expressed in the domestic currency. There are four of them which are : (1) the value $Y_t^{(1)} = S_t \exp\left(\int_0^t \delta_s ds\right)$ of a portfolio initially formed with the commodity S_0, and, whenever they are perceived, the profits are reinvested to buy more of the commodity; (2) the value $Y_t^{(2)} = C_t D_t^*$ of the foreign bank account expressed in the domestic currency; (3) the domestic zero-coupon bond; and (4) the value $Y^{(3)}(t,T) = C_t P^*(t,T)$ of the foreign zero-coupon bond converted in the domestic currency. Using the standard methodology, Q is constructed such that the four relevant assets have the risk free rate as return. Details are available from the authors upon request.

3 Hedging performance of the European basket option: a Monte Carlo study

Whenever the underlying assets are not perfectly and positively correlated, the portfolio is partially diversified and its volatility is then reduced. We can apply this reasoning to a basket option which gives to its owner the right to buy or sell the portfolio at a predetermined exercise price at a pre-specified date. Hence, the basket option allows to hedge simultaneously different financial risks such as the fluctuations of the commodity price, the exchange rate and the interest rates at a possible lower cost than the one associated to the individual hedge of each of these risks. The advantage link to the basket option should increase as the portfolio is well diversified, including assets with negative correlation. In this section, we will demonstrate numerically that the basket option is cheaper than a portfolio of standard options and analyze its hedging performance. However, this analysis does not account for the possible lack of liquidity of the basket option.

We consider an European basket option which gives to its holder the opportunity to sell, at time T and at the exercise price K_B, a portfolio formed with the commodity, a domestic zero-coupon bond (with maturity $T_1 \geq T$) and a foreign zero-coupon bond (with maturity $T_2 \geq T$) converted in domestic currency. We assume that w_1, w_2 and w_3 correspond respectively to the number of shares initially invested in the commodity, the domestic bond and the foreign bond. The time t value of this option is

$$V_t^{Basket} = D_t \mathrm{E}_t^Q \left[D_T^{-1} \max \left(K_B - w_1 S_T - w_2 P\left(T, T_1\right) - w_3 C_T P^*\left(T, T_2\right); 0 \right) \right].$$

Since the portfolio value is a weighted sum of lognormally distributed random variables, there is no closed form solution to this valuation problem[4] and the pricing is obtained via

[4] Dionne *et al.* (2008) proposed some analytical approximations to price an heterogeneous basket option.

Table 1: Parameters' distributions

Drift coefficients	Volatilities	Correlations
$\alpha_s \sim U(-0.10; +0.35)$	$\sigma_s \sim U(0.100; 0.25)$	$\rho_{12} \sim U(+0.05; +0.35)$
$\kappa \sim U(+0.05; +0.65)$	$\sigma_\delta \sim U(0.015; 0.045)$	$\rho_{13} \sim U(-0.40; +0.20)$
$\theta \sim U(+0.01; +0.35)$		$\rho_{14} \sim U(-0.40; +0.20)$
$\alpha_c \sim U(-0.04; +0.07)$	$\sigma_c \sim U(0.030; 0.10)$	$\rho_{23} \sim U(-0.30; +0.30)$
$f(0,t) = r \sim U(+0.01; +0.07)$	$\eta \sim U(0.001; 0.04)$	$\rho_{24} \sim U(-0.30; +0.30)$
$f^*(0,t) = r^* \sim U(+0.01; +0.07)$	$\eta^* \sim U(0.001; 0.04)$	$\rho_{34} \sim U(+0.35; +0.80)$
$\lambda \sim U(+0.01; +0.05)$		
$\widetilde{\lambda} \sim U(+0.01; +0.05)$		

$x \sim U(a; b)$ means that x has been simulated using a uniform distribution on the interval $[a, b]$. For this study, the volatility parameters of both instantaneous forward rates models is set to some constants, that is, for any $0 \leq t \leq T$, $\eta_{t,T} = \eta$ and $\eta^*_{t,T} = \eta^*$.

Monte Carlo simulations.

We now analyze empirically the performance of basket option as an hedging instrument. To avoid the possibility that the results may be influenced by the choice of model parameters, we compute option prices over a wide range of parameters. Like Broadie and Detemple (1996), we use 1,000 parameters' combinations generated randomly from a realistic set of values and assuming a continuous uniform distribution as presented in Table 1.

We consider a gold mining firm that, in six month from now ($T = 1/2$), will sell $w_1 = 10,000$ ounces of gold, sell $w_2 = 1,500,000$ domestic zero-coupon bonds (with maturity $T_1 = 3/4$) and convert $w_3 = 2,000,000$ of foreign currency in the domestic currency. To reduce its risk, this firm may choose between buying a basket put option or buying a portfolio of individual options. We assume that the firm holds the risky assets. The determination of the optimal composition of the basket that accounts for the correlations between the assets

are beyond the goals of this study.

Using the objective measure P and for each parameters set, $1,000$ scenarios of possible gold prices, exchange rates and domestic bond prices are simulated. For each generated scenario, the profit and the return of both hedging strategies are computed. More precisely, let

$$B_t = w_1 S_t + w_2 P(t, T_1) + w_3 C_t P^*(t, T)$$

be the time t value of the basket. Note that the foreign bond has the same maturity than the option. The profit

$$PR_B = \max(K_B - B_T, 0) + (B_T - B_0) - V_0^{Basket}$$

associated to the basket option strategy corresponds to the cash-flows generated by the basket option's exercise to which is added the profit (or loss) associated with the detention of the assets and minus the initial price V_0^{Basket} of the basket option. The profit

$$\begin{aligned} PR_{IO} &= \max[w_1(K_S - S_T), 0] + \max[w_2(K_P - P(T, T_1)), 0] + \max[w_3(K_C - C_T), 0] \\ &\quad + (B_T - B_0) - \left(V_0^{Gold} + V_0^{Bond} + V_0^{Fx}\right) \end{aligned}$$

associated to the individual options strategy is composed of the cash-flows generated at time T by the exercise of the put option on gold price, the put option on the domestic zero-coupon bond and the put option on the exchange rate to which is added the profit (or loss) associated with the detention of the assets and minus the initial prices V_0^{Gold}, V_0^{Bond}, V_0^{Fx} of the individual options. K_B, K_S, K_P and K_C correspond respectively to the exercise prices of the basket option, the gold price option, the domestic bond option and the exchange rate option. All option's prices are computed under the risk-free measure Q using a Monte Carlo

simulation with 200,000 trajectories and an antithetic variable. The profit associated with a non-hedging strategy is simply

$$PR_{NH} = B_T - B_0.$$

The returns of the basket option strategy, the individual options strategy and the non-hedging strategy are defined respectively by

$$RT_B = \frac{PR_B}{V_0^{Basket} + B_0},\ RT_{IO} = \frac{PR_{IO}}{V_0^{Gold} + V_0^{Bond} + V_0^{Fx} + B_0} \text{ and } RT_{NH} = \frac{PR_{NH}}{B_0}.$$

The exercise prices are determined to favor the exercise of each of the options. More precisely, each exercise price corresponds to some predetermined quantile of the underlying assets' prices at maturity date T. Technically, the 1,000 simulated prices are ordered and the exercise price is fixed such that it is larger than $i\%$ of the simulated prices.

For each parameter sets, the percentage ($\%_{PR}$) of the 1,000 scenarios for which the profit PR_B associated to the basket option strategy is larger than the profit PR_{IO} of the individual options strategy have been calculated.

At a first glance on Table 2, the non-hedging strategy seems to dominate the basket strategy since the profits associated to the basket strategy are larger than the non hedging strategy's ones in some moderate proportions ranging between 37% to 57% of the simulated scenarios. However, looking at the average profits and returns, the basket option strategy surpasses the non-hedging strategy. The introduction of the basket option shifts the portfolio distribution to the left (because of the initial cost) but increases the right tail of the distribution as the protection comes in (see Figure 1). The benefits of the basket option enlarge the right tail of the distribution much more significantly than the option price contributes to the left tail of the distribution. The asymmetric effect leads to the basket portfolio's

Table 2: Hedging performance of the basket option using random parameters

	$Q_{0.90}$	$Q_{0.80}$	$Q_{0.70}$	$Q_{0.60}$	$Q_{0.50}$
$\overline{\%}_{PR-NH}$	56,7%	52,2%	47,7%	42,8%	37,1%
$\overline{\%}_{PR-IO}$	95.1%	91.0%	87.8%	86.4%	85.2%
$\overline{\overline{PR}}_{NH}$	312 226 $	312 226 $	312 226 $	312 226 $	312 226 $
$\overline{\overline{PR}}_{B}$	500 811 $	459 996 $	428 513 $	402 272 $	378 569 $
$\overline{\overline{PR}}_{IO}$	460 090 $	418 571 $	388 277 $	360 674 $	338 745 $
Sharpe ratio$_{NH}$	0.46	0.46	0.46	0.46	0.46
Sharpe ratio$_{B}$	0.38	0.36	0.35	0.34	0.34
Sharpe ratio$_{IO}$	0.34	0.32	0.31	0.31	0.30
$\%_{RT-NH}$	56,1%	51,6%	47,1%	42,2%	36,6%
$\%_{RT-IO}$	94,4%	90,5%	87,9%	86,5%	85,8%
$\overline{\overline{RT}}_{NH}$	3,7%	3,7%	3,7%	3,7%	3,7%
$\overline{\overline{RT}}_{B}$	5,7%	5,3%	5,0%	4,7%	4,5%
$\overline{\overline{RT}}_{IO}$	5,2%	4,8%	4,5%	4,2%	4,0%
$\overline{V}_{B}$	667 484 $	481 854 $	369 683 $	286 766 $	218 152 $
$\overline{V}_{IO}$	800 867 $	586 443 $	461 844 $	361 241 $	286 913 $

1000 scenarios for each of the 1000 parameters sets have been simulated. Q_i means that the different exercise prices, K_B, K_S, K_P, and K_C, are set to the i^{th} quantile of $B_T, S_T, P(T, T_1)$, and C_T respectively (these Q_i vary with the parameter sets). The initial values are $S_0 = 325\$$, $C_0 = 0.85$, $\delta_0 = 1\%$. $\overline{\%}_{PR-\bullet}$ $\left(\overline{\%}_{RT-\bullet}\right)$ is the percentage of the 10^6 scenarios for which the profits (returns) associated to the basket option strategy is larger than the profits (returns) of the other strategy. $\overline{\overline{PR}}_{\bullet}$ $\left(\overline{\overline{RT}}_{\bullet}\right)$ are the average profits (returns) based on the 10^6 scenarios. The average Sharpe ratios over the 1000 parameter sets are reported. $\overline{V}_B$ and $\overline{V}_{IO}$ represent respectively the average basket option price and the sum of the average individual option prices.

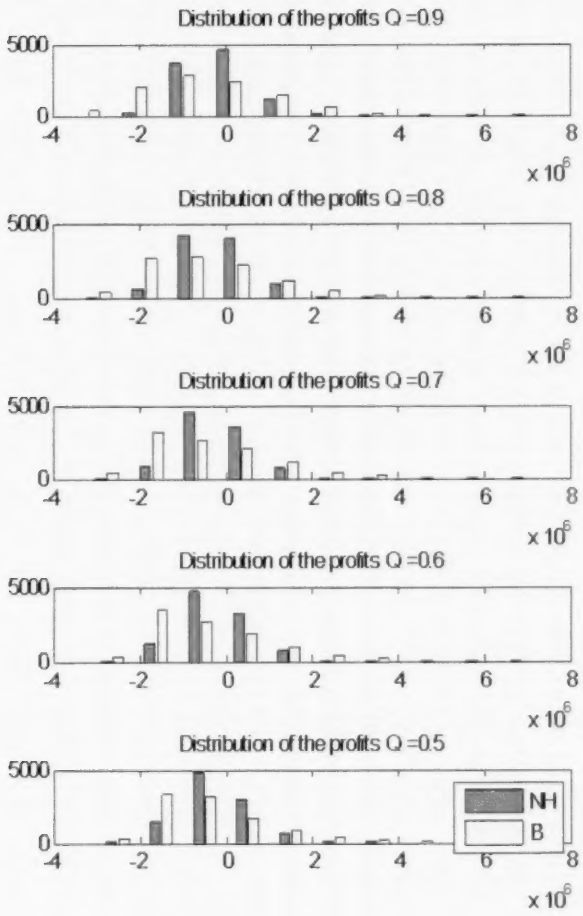

Figure 1: **Portfolio distributions for one of the 1000 simulated parameter sets.** NH stands for the non-hedging strategy while B represents the basket option strategy.

distribution which has a larger mean and a larger variance than the non-hedging strategy. Since the Sharp ratio only accounts for the two first moments of the distribution, it in not an appropriate risk measure in this context as it penalizes the right tail of the distribution as much as the left tail responsible for the losses.

Compared to the individual options strategy, the basket option dominates in each of the considered measures. However, there are a couple of other aspects of basket options that should be mentioned. In practice investment banks who issue these options tend to include high margins in their pricing since the contracts are difficult to hedge. Furthermore, some of them are very sensitive to the correlations and correlations are often unstable and difficult to estimate. This tends to increase their price.

4 Parameters estimation

In this section, the parameters of Model (1) are estimated using the maximum likelihood framework. However, this is not a straight forward application principally because the convenience yield is not an observable variable. More precisely, let $t_0 < t_1 < ... < t_n$ be the points in time where the sample is observed and note that $\ln\left(S_{t_i}/S_{t_{i-1}}\right)$ depends on the convenience yield $\delta_{t_{i-1}}$:

$$\begin{aligned}\ln\frac{S_{t_i}}{S_{t_{i-1}}} &= \left(\alpha_s - \frac{\sigma_s^2}{2} - \theta\right)(t_i - t_{i-1}) - \left(\delta_{t_{i-1}} - \theta\right)\frac{1-\exp\left(-\kappa\left(t_i - t_{i-1}\right)\right)}{\kappa} \\ &\quad + \int_{t_{i-1}}^{t_i}\left(\sigma_s - \sigma_\delta\frac{1-\exp\left(-\kappa\left(t_i - v\right)\right)}{\kappa}\right)dW_v^{(1)}.\end{aligned}$$

We rely on forward contracts on the commodity to estimate $\delta_{t_{i-1}}$. Let $F(t,T)$ denotes the time t value of a forward contract on the commodity with maturity date T. As shown in Appendix A, if the time to maturity $\varepsilon = T - t$ of the contract is small, then the convenience

yield may be approximated by:

$$\begin{aligned}\delta_{t_{i-1}} &\cong \frac{1}{\varepsilon}\ln\frac{S_{t_{i-1}}}{F\left(t_{i-1},t_{i-1}+\varepsilon\right)P\left(t_{i-1},t_{i-1}+\varepsilon\right)}+\frac{1}{2\varepsilon}\mathrm{Var}_{t_{i-1}}^{Q}\left[\ln\left(S_{t_{i-1}+\varepsilon}\right)\right]\\ &\quad-\frac{\sigma_s^2}{2}-\frac{1}{2\varepsilon}\int_{t_{i-1}}^{t_{i-1}+\varepsilon}\beta_{v,t_{i-1}+\varepsilon}^2 dv\end{aligned}$$

where $\mathrm{Var}_{t_{i-1}}^{Q}\left[\ln\left(S_{t_{i-1}+\varepsilon}\right)\right]$, given in the appendix (at line (5), page 29), is a function of time and the maturity date. Note that it is possible to find the exact expression for $\delta_{t_{i-1}}$ using the forward contract $F\left(t_{i-1},T_{i-1}\right)$ with an arbitrary maturity date T_{i-1} but it involves the instantaneous forward rates $f\left(t_{i-1},u\right)$, $t_{i-1}\leq u\leq T_{i-1}$ which would have to be estimated at each sampling date (for a sample of size n it requires the estimation of n term structures of instantaneous forward rates).

We now determine what should be the other assets to be observed at each sampling date. We argue that it is better to use the forward contracts on zero-coupon bonds instead of the bonds themselves. First, let consider the domestic case. Following HJM, there is a close relationship between the drift and the diffusion terms of the forward rates which is $\gamma_{t,T}=\eta_{t,T}\left(\beta_{t,T}+\lambda_t^{(3)}\right)$ where $\lambda^{(3)}$ is some risk premium and $\beta_{t,T}=\int_t^T\eta_{t,s}ds$. This relationship appears in the construction of the risk neutral measure Q. Since the domestic zero-coupon bond satisfies the relationship

$$\begin{aligned}\ln\frac{P\left(t_i,T\right)}{P\left(t_{i-1},T\right)} &= \int_{t_{i-1}}^{t_i}f\left(t_{i-1},u\right)du-\frac{1}{2}\int_{t_{i-1}}^{t_i}\left(\beta_{v,T}^2-\beta_{v,t_i}^2\right)dv\\ &\quad-\int_{t_{i-1}}^{t_i}\lambda_v^{(3)}\left(\int_{t_i}^{T}\eta_{v,u}du\right)dv-\int_{t_{i-1}}^{t_i}\left(\int_{t_i}^{T}\eta_{v,u}du\right)dW_v^{(3)},\end{aligned}$$

then the term structure of the instantaneous forward rates $f\left(t_{i-1},\bullet\right)$ is required at each sampling date. However, these rates are not directly observable and, to avoid their estimation, we rely on forward contracts on zero-coupon bonds. Indeed, if $F\left(t_i,T_i,U_i\right)$ denotes the time t_i value of some forward contract on a zero-coupon bond, where T_i is the maturity date of the contract and $U_i>T_i$ is the maturity date of the underlying zero-coupon bond, then[5] for

[5] **Sketch of the proof**. Following Jarrow (1996), the $F(\bullet,T,U)$ is a $Q-$martingale. Therefore,

$0 \leq t_{i-1} \leq t_i \leq T_i \leq U_i,$

$$\ln \frac{F(t_i, T_i, U_i)}{F(t_{i-1}, T_i, U_i)} = \begin{pmatrix} -\frac{1}{2}\int_{t_{i-1}}^{t_i} \left(\beta_{u,T_i} - \beta_{u,U_i}\right)^2 du \\ -\int_{t_{i-1}}^{t_i} \left(\int_{T_i}^{U_i} \eta_{v,u} du\right) \lambda_v^{(3)} dv - \int_{t_{i-1}}^{t_i} \left(\int_{T_i}^{U_i} \eta_{v,u} du\right) dW_v^{(3)} \end{pmatrix}.$$

Therefore, if the risk premium is a deterministic function of time t, then $\{F(t, T_i, U_i) : 0 \leq t \leq T_i\}$ is a Gaussian Markovian process under the objective measure P and depends only on the diffusion coefficient $\eta_{\bullet,\bullet}$ and the risk premium $\lambda_\bullet^{(3)}$.

Similarly, the case of the foreign bond is as follows: the relationship between the drift and the diffusion terms of the forward rates is $\gamma_{t,T}^* = \eta_{t,T}^* \left(\beta_{t,T}^* - \sigma_c \rho_{2,4} + \lambda_t^{(4)}\right)$ where $\lambda^{(4)}$ is a risk premium and $\beta_{t,T}^* = \int_t^T \eta_{t,s}^* ds$. The foreign zero-coupon bond requires the unobserved term structure of the instantaneous forward rates $f^*(t, \bullet)$. Let $F^*(t, T_i^*, U_i^*)$ denotes the time t value of some forward contracts on a foreign zero-coupon bond, where T_i^* is the maturity date of the contract and $U_i^* > T_i^*$ is the maturity date of the underlying zero-coupon bond. The forward contract value satifies

$$\ln \frac{F^*(t_i, T_i^*, U_i^*)}{F^*(t_{i-1}, T_i^*, U_i^*)} = \begin{pmatrix} -\frac{1}{2}\int_{t_{i-1}}^{t_i} \left(\beta_{u,U_i^*}^* - \beta_{u,T_i^*}^*\right)^2 du + \sigma_c \rho_{2,4} \int_{t_{i-1}}^{t_i} \left(\int_{T_i^*}^{U_i^*} \eta_{v,u}^* du\right) dv \\ -\int_{t_{i-1}}^{t_i} \lambda_v^{(4)} \left(\int_{T_i^*}^{U_i^*} \eta_{v,u}^* du\right) dv - \int_{t_{i-1}}^{t_i} \left(\int_{T_i^*}^{U_i^*} \eta_{v,u}^* du\right) dW_v^{(4)} \end{pmatrix}.$$

Consequently, if the risk premium is a deterministic function of time t, then the stochastic process $\{F^*(t, T_i^*, U_i^*) : 0 \leq t \leq T^*\}$ is Markovian and normally distributed under the objective measure P and depends only on the diffusion coefficients $\eta_{\bullet,\bullet}^*$ and σ_c, the correlation coefficient $\rho_{2,4}$ and the risk premium $\lambda_\bullet^{(4)}$.

The last component of the sample is based on the exchange rate

$$\ln \frac{C_{t_i}}{C_{t_{i-1}}} = \left(\alpha_C - \frac{\sigma_C^2}{2}\right)(t_i - t_{i-1}) + \sigma_C \left(W_{t_i}^{(2)} - W_{t_{i-1}}^{(2)}\right).$$

$F(t,T,U) = \mathrm{E}_t^Q[F(T,T,U)] = \mathrm{E}_t^Q[P(T,U)] = \exp\left(\mathrm{E}_t^Q[\ln P(T,U)] + \frac{1}{2}\mathrm{Var}_t^Q[\ln P(T,U)]\right)$ where $\mathrm{E}_t^Q[\bullet]$ denotes the conditional expectation with respect to the information available at time t : $\mathrm{E}^Q[\bullet|\mathcal{F}_t]$. The last equality is justified by the lognormal distribution of $P(T,U)$. The final result is obtained from the evaluation of the conditional moments of $\ln P(T,U)$ under the risk neutral measure Q.

For pragmatic reasons and to keep the number of parameters to be estimated as small as possible, we have set the risk premiums to be constant: for any $t \geq 0$, $\lambda_t^{(3)} = \lambda$ and $\lambda_t^{(4)} = \lambda^*$. Let $\boldsymbol{\pi} = (\boldsymbol{\pi}_1, \boldsymbol{\pi}_2)$ denotes the set of parameters that will be estimated where $\boldsymbol{\pi}_1$ contains the parameters needed in the pricing of the basket option while $\boldsymbol{\pi}_2$ are some parameters that will be estimated but not used in the pricing procedure:

$$\begin{aligned} \boldsymbol{\pi}_1 &= \left(\kappa, \theta, \sigma_s, \sigma_\delta, \sigma_c, \eta_{t,T}, \eta_{t,T}^*, \rho_{1,2}, \rho_{1,3}, \rho_{1,4}, \rho_{2,3}, \rho_{2,4}, \rho_{3,4}\right), \\ \boldsymbol{\pi}_2 &= \left(\alpha_s, \alpha_c, \lambda, \lambda^*\right). \end{aligned}$$

Define

$$\mathbf{X}_{t_i} = \begin{pmatrix} \ln\left(S_{t_i}/S_{t_{i-1}}\right) \\ \ln\left(C_{t_i}/C_{t_{i-1}}\right) \\ \ln\left(F\left(t_i, T_i, U_i\right)/F\left(t_{i-1}, T_i, U_i\right)\right) \\ \ln\left(F^*\left(t_i, T_i^*, U_i^*\right)/F^*\left(t_{i-1}, T_i^*, U_i^*\right)\right) \end{pmatrix}.$$

As shown in Appendix B, the log-likelihood function associated with the observed sample $\mathbf{x}_{t_1}, ..., \mathbf{x}_{t_n}$ is

$$\mathcal{L}\left(\boldsymbol{\pi}_1, \boldsymbol{\pi}_2; \mathbf{x}_{t_1}, ...\mathbf{x}_{t_n}\right) = -2n\ln\left(2\pi\right) - \frac{1}{2}\sum_{i=1}^{n}\ln\left|\boldsymbol{\Sigma}_{t_i}\right| - \frac{1}{2}\sum_{i=1}^{n}\left(\mathbf{x}_{t_i} - \boldsymbol{\mu}_{t_i}\right)'\boldsymbol{\Sigma}_{t_i}^{-1}\left(\mathbf{x}_{t_i} - \boldsymbol{\mu}_{t_i}\right), \quad (4)$$

where $\boldsymbol{\mu}_{t_i} = \mathrm{E}_{t_{i-1}}\left[\mathbf{X}_{t_i}\right]$ and covariance matrix $\boldsymbol{\Sigma}_{t_i} = \mathrm{Var}_{t_{i-1}}\left[\mathbf{X}_{t_i}\right]$ are given in Appendix B.

Because of the large number of parameters to be estimated, it is difficult to maximize Equation (4) directly. We therefore follow a two steps procedure:

Step 1 : We estimate the parameters $\boldsymbol{\pi}_3 = \left(\alpha_c, \sigma_c, \lambda, \lambda^*, \eta_{t,T}, \eta_{t,T}^*, \rho_{2,3}, \rho_{2,4}, \rho_{3,4}\right)$ associated to the exchange rate and the domestic and foreign interest rates using the log-likelihood function

$$\mathcal{L}^*\left(\boldsymbol{\pi}_3, \mathbf{z}_{t_1}, ...\mathbf{z}_{t_n}\right) = -2n\ln\left(2\pi\right) - \frac{1}{2}\sum_{i=1}^{n}\ln\left|\boldsymbol{\Gamma}_{t_i}\right| - \frac{1}{2}\sum_{i=1}^{n}\left(\mathbf{z}_{t_i} - \boldsymbol{\mu}_{t_i}^*\right)'\boldsymbol{\Gamma}_{t_i}^{-1}\left(\mathbf{z}_{t_i} - \boldsymbol{\mu}_{t_i}^*\right),$$

where $\mathbf{z}_{t_i}$ contains the three last components of $\mathbf{x}_{t_i}$, $\boldsymbol{\mu}_{t_i}^*$ is formed with the three last components of $\boldsymbol{\mu}_{t_i}$ and $\boldsymbol{\Gamma}_{t_i}$ is the 3×3 matrix $\left(\sigma_{t_i,\ell,j}\right)_{\ell,j=2,3,4}$.

Note that in the case[6] where $\boldsymbol{\mu}_{t_i}^* = \boldsymbol{\mu}^*$ and $\boldsymbol{\Gamma}_{t_i} = \boldsymbol{\Gamma}$, that is, the two first moments are constant through time, then it is possible to find analytically the maximum likelihood estimates $\widehat{\boldsymbol{\mu}}^*$ and $\widehat{\boldsymbol{\Gamma}}$ that maximize the log-likelihood function $\mathcal{L}^*$. The parameters estimates $\widehat{\boldsymbol{\pi}}_3$ are chosen such that $\boldsymbol{\mu}^*(\widehat{\boldsymbol{\pi}}_3) = \widehat{\boldsymbol{\mu}}^*$ and $\boldsymbol{\Gamma}(\widehat{\boldsymbol{\pi}}_3) = \widehat{\boldsymbol{\Gamma}}$.

Step 2 : Assuming that $\boldsymbol{\pi}_3 = \widehat{\boldsymbol{\pi}}_3$, then the log-likelihood $\mathcal{L}(\boldsymbol{\pi}_1, \boldsymbol{\pi}_2; \mathbf{x}_{t_1}, ...\mathbf{x}_{t_n})$ is maximized to get estimates for $\alpha_s, \kappa, \theta, \sigma_s, \sigma_\delta, \rho_{1,2}, \rho_{1,3}$ and $\rho_{1,4}$.

The numerical optimization routine used to maximize these two log-likelihood functions is the quadratic hill-climbing algorithm of Goldfeld, Quandt and Trotter (1996) with a convergence criterion based on the absolute values of the variations in parameter values and functional values between successive iterations. When both of these changes are smaller than 10^{-5}, we attend convergence. The solution obtained this way may not maximize the global loglikelihood function (4). We therefore perform a Monte Carlo study to assess numerically the quality of our estimates.

4.1 Monte Carlo study

We conduct a Monte Carlo study to evaluate the quality of the coefficients estimated using the maximum likelihood method. We verify numerically that the two-step procedure do not produce biased estimates. Moreover, we assess numerically how well the asymptotic normal distribution proposed by the theory approximates the empirical distributions for a reasonable sample size. More precisely, we generate daily observations for two different sampling periods: 4 and 10 years. For each time series, maximum likelihood estimates

[6] In the particular case where the volatility parameters $\eta_{t,T} = \eta$ and $\eta_{t,T}^* = \eta^*$ of the instantaneous forward rates are constant, the time between to sample observations $t_i - t_{i-1} = h$ is constant, and the differences between the maturity date of the underlying bond and the maturity date of the forward contract $U_i - T_i = H$ and $U_i^* - T_i^* = H^*$ are constant, then

$$\boldsymbol{\mu}^* = \begin{pmatrix} \left(\alpha_C - \frac{\sigma_C^2}{2}\right)h \\ -\eta h H\left(\lambda + \frac{1}{2}\eta H\right) \\ -\eta^* h H^*\left(\lambda^* + \frac{1}{2}\eta^* H^* - \sigma_c \rho_{2,4}\right) \end{pmatrix} \text{ and } \boldsymbol{\Gamma} = \begin{pmatrix} \sigma_C^2 h & -\rho_{2,3}\sigma_C \eta H h & -\rho_{2,4}\sigma_C \eta^* H^* h \\ -\rho_{2,3}\sigma_C \eta H h & \eta^2 H^2 h & \rho_{3,4}\eta\eta^* H H^* h \\ -\rho_{2,4}\sigma_C \eta^* H^* h & \rho_{3,4}\eta\eta^* H H^* h & (\eta^*)^2 (H^*)^2 h \end{pmatrix}.$$

Table 3: Simulations' results for the parameters estimations (4 years sample)

	α_c	σ_c	λ	λ^*	η	η^*	ρ_{23}	ρ_{24}	ρ_{34}
True	0.0900	0.0300	1.2000	1.2000	0.0200	0.0150	0.1500	0.2000	0.8500
Mean	0.0911	0.0299	1.2589	1.2649	0.0199	0.0149	0.1501	0.2010	0.8497
Median	0.0911	0.0299	1.2325	1.2515	0.0200	0.0150	0.1515	0.2016	0.8502
Std	0.0218	0.0013	0.6215	0.6098	0.0016	0.0022	0.0538	0.0464	0.0127
25 % cvr	0.2460	0.2405	**0.2980**	**0.3115**	0.2655	0.2695	0.2510	0.2745	0.2450
50 % cvr	0.4910	0.4960	**0.5315**	**0.5465**	0.5090	0.5065	0.4930	0.5085	0.5000
75 % cvr	0.7435	0.7435	0.7470	0.7515	0.7520	0.7525	0.7625	0.7475	0.7590
90 % cvr	0.8910	0.9000	**0.8575**	**0.8600**	0.8990	0.9035	0.9080	0.9060	0.8970
95 % cvr	0.9430	0.9490	**0.8930**	**0.8965**	0.9450	0.9475	0.9480	0.9445	0.9415
99 % cvr	0.9850	0.9880	**0.9315**	**0.9365**	**0.9835**	0.9850	**0.9830**	**0.9810**	**0.9820**

Mean, median and std are the descriptive statistics based on the simulated sample of 2000 parameter's estimates. The coverage rates (cvr) represent the proportion of the confidence intervals based on the Gaussian distribution that contain the true parameter's value. The estimates written in bold are significantly different than their theoritical counterpart (at a confidence level of 95%).

are computed as well as their associated estimated standard error and confidence intervals based on the Gaussian distribution. We repeat the simulation run 2000 times and reports averages of the points estimates and the proportions of the simulated scenarios producing confidence intervals that contain the true parameter value. If the Gaussian distribution and the estimation of the standard error are appropriate, then the proportions should be close to their corresponding confidence level. The forward contracts on the commodity have a time-to-maturity of 1 day.

As it appears in Tables 3, 4, 5 and 6, for all the parameters, the maximum likelihood estimators are unbiased. However, the standard deviations of the risk premium estimators as well as the convenience yield's parameters are large, which means that the punctual estimation is imprecise. The coverage rate associated to these parameters indicates that the asymptotic distribution has not been reached, even with the 10 years sample. For all other parameters, the standard errors indicate that we are in presence of precise punctual

Table 4: Simulations' results for the parameters estimations (4 years sample)

	α_s	σ_s	κ	θ	σ_δ	ρ_{12}	ρ_{13}	ρ_{14}
True	0.2500	0.1200	0.2000	0.1000	0.1500	-0.1000	-0.2500	-0.3000
Mean	0.2715	0.1197	0.1605	0.1089	0.1676	-0.0976	-0.2493	-0.2997
Median	0.2677	0.1197	0.1622	0.1058	0.1260	-0.0991	-0.2499	-0.3008
Std	0.0871	0.0043	0.1337	0.0578	0.1831	0.0498	0.0412	0.0405
25 % cvr	0.2590	0.2535	**0.4420**	**0.6695**	**0.2135**	0.2505	0.2600	0.2440
50 % cvr	0.5090	0.4935	**0.5385**	**0.8300**	**0.4065**	0.4815	0.4860	0.4960
75 % cvr	0.7450	0.7390	**0.5815**	**0.8965**	**0.4870**	0.7375	0.7410	0.7505
90 % cvr	0.8875	0.8920	**0.6120**	**0.9325**	**0.5255**	**0.8640**	0.8955	0.8865
95 % cvr	**0.9345**	0.9460	**0.6255**	0.9435	**0.5345**	**0.9080**	0.9425	**0.9315**
99 % cvr	**0.9735**	0.9860	**0.6475**	**0.9525**	**0.5500**	**0.9445**	**0.9835**	0.9850

Mean, median and std are the descriptive statistics based on the simulated sample of 2000 parameter's estimates. The coverage rates (cvr) represent the proportion of the confidence intervals based on the Gaussian distribution that contain the true parameter's value. The estimates written in bold are significantly different than their theoritical counterpart (at a confidence level of 95%).

estimators and that the Gaussian distribution is appropriate for inference.

4.2 Real data study

In the following, we apply the procedure outlined in Section 4 to real data.

4.3 Data

In order to estimate the parameters of the domestic and foreign zero-coupon bonds $\hat{\eta}$, $\hat{\eta^*}$ and $\hat{\rho}_{34}$, we use the three-month Eurodollar Time Deposit futures contracts traded on the Chicago Mercantile Exchange (CME), and the three-month Canadian Bankers' Acceptance (BAX) futures contracts traded in the Montreal Exchange[7]. Both BAX and Eurodollar futures contracts are settled in cash and have the same delivery date on the second London bank business day immediately preceding the third Wednesday of the contract month. While the

[7]Both the Eurodollar Time Deposit and the BAX are on a $1 million principal value with a maturity of 90 days.

Table 5: Simulations' results for the parameters estimations (10 years sample)

	α_c	σ_c	λ	λ^*	η	η^*	ρ_{23}	ρ_{24}	ρ_{34}
True	0.0900	0.0300	1.2000	1.2000	0.0200	0.0150	0.1500	0.2000	0.8500
Mean	0.0898	0.0300	1.1986	1.1994	0.0200	0.0150	0.1499	0.1998	0.8499
Median	0.0900	0.0300	1.2087	1.1941	0.0200	0.0150	0.1503	0.2000	0.8501
Std	0.0093	0.0004	0.3136	0.3209	0.0003	0.0002	0.0186	0.0185	0.0055
25 % cvr	0.2618	0.2495	**0.2811**	**0.2781**	0.2663	0.2623	**0.2825**	**0.2781**	0.2505
50 % cvr	0.4985	0.5030	0.5148	0.5059	0.5118	0.5049	0.5286	**0.5350**	0.5108
75 % cvr	0.7456	0.7392	**0.7130**	**0.7106**	0.7510	0.7623	0.7766	**0.7771**	0.7579
90 % cvr	0.9078	0.8955	**0.8402**	**0.8269**	0.8881	0.9073	0.9083	0.8999	0.9019
95 % cvr	0.9443	0.9487	**0.8802**	**0.8787**	0.9487	0.9551	0.9443	0.9433	0.9512
99 % cvr	**0.9768**	0.9882	**0.9285**	**0.9334**	0.9852	0.9877	**0.9808**	**0.9798**	0.9887

Mean, median and std are the descriptive statistics based on the simulated sample of 2000 parameter's estimates. The coverage rates (cvr) represent the proportion of the confidence intervals based on the Gaussian distribution that contain the true parameter's value. The estimates written in bold are significantly different than their theoritical counterpart (at a confidence level of 95%).

Eurodollar contract is chosen for its extreme liquidity, the less liquid BAX contract represents the more tradable contract on a riskless zero-coupon bond available in Canada. Our sample consists of daily prices for both contracts ranging from January 3, 2005 to December 29, 2006. It should be noted that both futures contracts are traded on an index basis that is the contract price is calculated by subtracting the annualized implied yield on the underlying from 100. For example, a December BAX (Eurodollar) contract quoted as 97.30 on the exchange floor implies a 2.70% (i.e. $100 - 97.30$) annual yield for the BAX (Eurodollar) issued in December. To carry out the estimation, we need the futures prices under the physical measure P; hence we must convert the quoted prices using the equations :

$$F = 1 - 0.25\left(1 - \frac{Z}{100}\right) \text{ and } F^* = \frac{1}{1 + \left(1 - \frac{Z^*}{100}\right)\frac{90}{360}},$$

where Z and Z^* represent the quoted price for the Eurodollar and BAX futures contracts respectively.

We also use daily gold prices as well as the CAD/USD exchange rate covering the same

Table 6: Simulations' results for the parameters estimations (10 years sample)

	α_s	σ_s	κ	θ	σ_δ	ρ_{12}	ρ_{13}	ρ_{14}
True	0.2500	0.1200	0.2000	0.1000	0.1500	-0.1000	-0.2500	-0.3000
Mean	0.2527	0.1199	0.1930	0.0977	0.1400	-0.1006	-0.2500	-0.2999
Median	0.2528	0.1199	0.1899	0.0977	0.1448	-0.1010	-0.2497	-0.3004
Std	0.0528	0.0017	0.0486	0.0437	0.0532	0.0194	0.0187	0.0184
25 % cvr	0.2489	0.2696	**0.4634**	**0.4643**	**0.2873**	0.2573	0.2602	0.2420
50 % cvr	0.4786	0.5224	**0.6916**	**0.6950**	0.4771	0.5027	0.5007	0.4988
75 % cvr	**0.7004**	0.7560	**0.8190**	**0.8077**	**0.6449**	0.7368	0.7304	0.7290
90 % cvr	**0.8396**	0.8977	0.8819	**0.8603**	**0.7467**	**0.8770**	0.8849	0.8869
95 % cvr	**0.8859**	0.9474	**0.9026**	**0.8844**	**0.7821**	**0.9188**	**0.9356**	0.9385
99 % cvr	**0.9415**	0.9902	**0.9297**	**0.9051**	**0.8259**	**0.9661**	**0.9818**	0.9848

Mean, median and std are the descriptive statistics based on the simulated sample of 2000 parameter's estimates. The coverage rates (cvr) represent the proportion of the confidence intervals based on the Gaussian distribution that contain the true parameter's value. The estimates written in bold are significantly different than their theoritical counterpart (at a confidence level of 95%).

time period as above to estimate the parameters related to the commodity price and the exchange rate: the drifs α_s and α_c, the volatilities σ_s and σ_c and the correlation coefficient ρ_{12}. Finally, we use gold futures contracts to estimate the convenience yield and its parameters.

The data is obtained from Datastream. Table 7 shows the summary statistics for the various data used.

4.4 Empirical results

We proceed with a two-step estimation in order to avoid any convergence problems. First, we estimate the exchange rate, the domestic and foreign zero-coupon bonds parameters as well as the correlation coefficients between these three variables $\left(\alpha_c, \sigma_c, \lambda, \lambda^*, \eta_{t,T}, \eta^*_{t,T}, \rho_{2,3}, \rho_{2,4}, \rho_{3,4}\right)$. Then, we use these estimates to determine the parameters related to the commodity and the convenience yield $\left(\alpha_s, \kappa, \theta, \sigma_s, \sigma_\delta, \widehat{\rho}_{1,2}, \rho_{1,3}, \rho_{1,4}\right)$ that maximize the global likelihood function given in Equation (4). We apply the quadratic hill-climbing algorithm of Goldfeld, Quandt

Table 7: Summary Statistics for daily observations between 01/03/2005 and 12/29/2006

Assets	Mean	Median	Standard deviation
Gold Prices	524.86	528.1	86.611
Exchange rates CAD/USD	0.854	0.858	0.033
Eurodollar futures contracts	0.989	0.988	0.002
BAX contracts	0.991	0.991	0.002
Gold futures contracts	525.57	527.9	86.659

Number of observations = 520

The descriptive statistics are based on a sample of daily observations over two years.

and Trotter (1996), and we use different starting points to increase the probability of reaching a global maxima[8]. The results from the maximum likelihood estimation are reported in Table 8.

The results show that the instantaneous returns' estimate of the commodity α_s and the exchange rate α_e are rather imprecise and statistically insignificant from zero. However, these two parameters are not used in the pricing of the basket option. The convenience yield mean reversion parameters' estimates, $\hat{\kappa} = 0.244$ and $\hat{\theta} = 0.276$, also are insignificant. This result tallies with the finding in Schwartz (1997) that mean reversion for convenience yields does not seem to hold for gold . However it may also be attributed to the small sample effect on these estimators. On the other hand, the volatility parameters for commodity, exchange rate, domestic and foreign zero-coupon bonds are estimated fairly accurately and are highly significantly different from zero.

The correlation coefficients are between the different Brownian motions. However, from the nature of the model, they are also related to the correlations among the logarithm of the forward contracts, gold prices and exchange rate as established in Appendix B. As expected, the correlation ρ_{34} between the Canadian and American zero-coupon bonds' noise terms is high and statistically different from zero. We observe a non-significant negative correlation between the Brownian motions involved in the gold prices and futures contracts on both Eu-

[8]Given that we have several parameters to estimate simultaneously, and that the algorithm is time-consuming, we choose only three different starting values for each parameter. For each repetition, we find that the algorithm converges to the same optima.

Table 8: Summary Statistics for daily observations between 01/03/2005 and 12/29/2006

	Estimate	Std	P-value		Estimate	Std	P-value
α_s	-0.1204	0.1056	0.255	κ	0.2440	0.1713	0.155
α_c	0.0208	0.0512	0.685	θ	0.2760	0.1765	0.119
σ_s	0.1664	0.0051	0.000	σ_δ	0.0151	0.0238	0.525
σ_c	0.0747	0.0023	0.000	ρ_{12}	0.3485	0.0373	0.000
η	0.0032	0.0001	0.000	ρ_{13}	-0.0594	0.0325	0.068
η^*	0.0031	0.0001	0.000	ρ_{14}	-0.0524	0.0363	0.150
λ	2.9313	0.7066	0.000	ρ_{23}	-0.0391	0.0431	0.366
λ^*	2.0211	0.7165	0.005	ρ_{24}	0.0454	0.0512	0.376
				ρ_{34}	0.5218	0.0319	0.000

The estimates and their standard deviations are obtained from daily observations over a two year sample, going from January 3, 2005 to December 29, 2006. The p-values are computed using the Gaussian distribution. These p-values should be interpreted with precaution since, for some parameters like the convenience yield's ones and the risk premia, the sample is to small to justify the use of the estimator's asymptotic distribution.

rodollars, $\hat{\rho}_{13} = -5.94\%$, and BAX, $\hat{\rho}_{14} = -5.24\%$. We also note a highly significant positive correlation between gold price and foreign exchange $\hat{\rho}_{12} = 34.85\%$. However, the correlations between the exchange rate and futures contracts on both domestic $\left(\hat{\rho}_{23} = -3.91\%\right)$ and foreign $\left(\hat{\rho}_{24} = 4.54\%\right)$ bonds are not significantly different from zero.

5 Conclusion

In this paper we develop a theoretical model for an heterogeneous basket option based on commodity prices, exchange rates, and zero-coupon bonds. Our contributions consist essentially in looking at a basket option based on multiple underlying assets which are intrinsically different and in considering all the aspects of basket options: modelization, performance analysis, and parameters' estimation. The empirical implementation of our model raises several problems. Many of the variables prove to be unobservable, variables such as the commodity's convenience yield, the market price of convenience-yield risk; and the market-price risk related to zero-coupon bonds. To overcome these problems, we first suppose that the process describing the convenience yield shares the same source of risk as

the commodity process; this simplification frees us from having to estimate the market-price risk related to the convenience yield. Second, we view the futures contract as a derivative instrument based on the instantaneous forward rate and so deriving its uncertainty from the same source of risk as the forward rate.

We make an empirical comparison between the performance of a basket-option strategy and that of a portfolio of individual plain-vanilla options using a large variety of parameters' values. Our results show that the heterogeneous basket option dominates the individual option strategy. Compared to the non-hedging strategy, the profits distribution has fatter tails with a positive skewness, meaning that the probability of larger profits is augmented. Consequently, the basket option is a good hedging strategy.

We estimate our theoretical model empirically, using both simulated and real data. Indeed, we apply the maximum-likelihood method to estimate the parameters of risky assets and we obtain satisfactory results.

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A Forward contract on commodity

Following Shreve (2004), the time t value of a forward contract with maturity date T is $F(t,T) = \mathrm{E}_t^Q[S_T]$. Therefore,

$$F(t,T) = \mathrm{E}_t^Q[S_T] = \mathrm{E}_t^Q[\exp(\ln(S_T))] = \exp\left(\mathrm{E}_t^Q[\ln(S_T)] + \frac{1}{2}\mathrm{Var}_t^Q[\ln(S_T)]\right)$$

since $\ln(S_T)$ is normally distributed under the measure Q and where $\mathrm{E}_t^Q[\bullet] = \mathrm{E}_t^Q[\bullet|\mathcal{F}_t]$. Recall that

$$\ln S_T = \ln S_t - \frac{\sigma_s^2}{2}(T-t) + \int_t^T r_u du - \int_t^T \delta_u du + \sigma_s \int_t^T d\widetilde{W}_u^{(1)}.$$

We first compute $\mathrm{E}_t^Q[\ln(S_T)] = \ln S_t - \frac{\sigma_s^2}{2}(T-t) - \mathrm{E}_t^Q\left[-\int_t^T r_u du\right] - \mathrm{E}_t^Q\left[\int_t^T \delta_u du\right]$. Since

$$\int_t^T r_u du = \int_t^T f(t,u)\,du + \int_t^T \left(\int_t^u \eta_{v,u}\beta_{v,u}dv\right)du + \int_t^T \beta_{v,T}d\widetilde{W}_v^{(3)}$$

and because any normally distributed random variable Z, $\mathrm{E}[\exp(Z)] = \exp\left(\mathrm{E}[Z] + \frac{1}{2}\mathrm{Var}[Z]\right)$ implies that $\mathrm{E}[Z] = \ln \mathrm{E}[\exp(Z)] - \frac{1}{2}\mathrm{Var}[Z]$,

$$\mathrm{E}_t^Q\left[-\int_t^T r_u du\right] = \ln \mathrm{E}_t^Q\left[\exp\left(-\int_t^T r_u du\right)\right] - \frac{1}{2}\mathrm{Var}_t^Q\left[\int_t^T r_u du\right] = \ln P(t,T) + \frac{1}{2}\int_t^T \beta_{v,T}^2 dv.$$

Moreover, if $T-t$ is small, $\int_t^T \delta_u du \cong \delta_t(T-t)$ and

$$\mathrm{E}_t^Q[\ln(S_T)] \cong \ln S_t - \frac{\sigma_s^2}{2}(T-t) - \ln P(t,T) - \frac{1}{2}\int_t^T \beta_{v,T}^2 dv - \delta_t(T-t).$$

We use this approximation to avoid the introduction of the instantaneous forward rates in the expression of $\mathrm{E}_t^Q[\ln(S_T)]$. Second, we evaluate $\mathrm{Var}_t^Q[\ln(S_T)]$. Since

$$\begin{aligned}
\ln S_T &= \ln S_t - \frac{\sigma_s^2}{2}(T-t) + \int_t^T r_u du - \int_t^T \delta_u du + \sigma_s \int_t^T d\widetilde{W}_u^{(1)}, \\
\int_t^T r_u du &= \int_t^T f(t,u)\,du + \int_t^T \left(\int_t^u \eta_{v,u}\beta_{v,u}dv\right)du + \int_t^T \beta_{v,T}d\widetilde{W}_v^{(3)}, \\
\int_t^T \delta_u du &= \left(\theta - \frac{\sigma_\delta}{\sigma_s}\frac{\alpha_s}{\kappa}\right)(T-t) + \left(\delta_t - \theta + \frac{\sigma_\delta}{\sigma_s}\frac{\alpha_s}{\kappa}\right)\frac{1-\exp(-\kappa(T-t))}{\kappa} \\
&\quad + \frac{\sigma_\delta}{\sigma_s}\int_t^T \frac{1-\exp(-\kappa(T-u))}{\kappa} f(t,u)\,du + \frac{\sigma_\delta}{\sigma_s}\int_t^T \frac{1-\exp(-\kappa(T-u))}{\kappa}\left(\int_t^u \eta_{v,u}\beta_{v,u}dv\right)du \\
&\quad + \frac{\sigma_\delta}{\sigma_s}\int_t^T \left(\int_v^T \eta_{v,u}\frac{1-\exp(-\kappa(T-u))}{\kappa}du\right)d\widetilde{W}_v^{(3)} + \sigma_\delta \int_t^T \frac{1-\exp(-\kappa(T-u))}{\kappa}d\widetilde{W}_u^{(1)},
\end{aligned}$$

then

$$\begin{aligned}
&\mathrm{Var}_t^Q\left[\ln\left(S_T\right)\right] \\
&= \int_t^T\left(\beta_{v,T}-\frac{\sigma_\delta}{\sigma_s}\int_v^T\eta_{v,u}\frac{1-\exp\left(-\kappa\left(T-u\right)\right)}{\kappa}du\right)^2dv+\int_t^T\left(\sigma_s-\sigma_\delta\frac{1-\exp\left(-\kappa\left(T-v\right)\right)}{\kappa}\right)^2dv \\
&\quad+2\rho_{13}\int_t^T\left(\sigma_s-\sigma_\delta\frac{1-\exp\left(-\kappa\left(T-v\right)\right)}{\kappa}\right)\left(\beta_{v,T}-\frac{\sigma_\delta}{\sigma_s}\int_v^T\eta_{v,u}\frac{1-\exp\left(-\kappa\left(T-u\right)\right)}{\kappa}du\right)dv \\
&= \int_t^T\left(\beta_{v,T}-\frac{\sigma_\delta}{\sigma_s}\int_v^T\eta_{v,u}\frac{1-\exp\left(-\kappa\left(T-u\right)\right)}{\kappa}du\right)^2dv \qquad (5)\\
&\quad+\left(\sigma_s-\frac{\sigma_\delta}{\kappa}\right)^2\left(T-t\right)+2\frac{\sigma_\delta}{\kappa}\left(\sigma_s-\frac{\sigma_\delta}{\kappa}\right)\frac{1-\exp\left(-\kappa\left(T-t\right)\right)}{\kappa}+\frac{\sigma_\delta^2}{\kappa^2}\frac{1-\exp\left(-2\kappa\left(T-t\right)\right)}{2\kappa} \\
&\quad+2\rho_{13}\int_t^T\left(\sigma_s-\sigma_\delta\frac{1-\exp\left(-\kappa\left(T-v\right)\right)}{\kappa}\right)\left(\beta_{v,T}-\frac{\sigma_\delta}{\sigma_s}\int_v^T\eta_{v,u}\frac{1-\exp\left(-\kappa\left(T-u\right)\right)}{\kappa}du\right)dv.
\end{aligned}$$

Hence,

$$F\left(t,T\right)\cong\exp\left(\ln S_t-\frac{\sigma_s^2}{2}\left(T-t\right)-\ln P\left(t,T\right)-\frac{1}{2}\int_t^T\beta_{v,T}^2dv-\delta_t\left(T-t\right)+\frac{1}{2}\mathrm{Var}_t^Q\left[\ln\left(S_T\right)\right]\right)$$

which implies that whenever $T-t$ is small,

$$\delta_t\cong\frac{\ln\frac{S_t}{F(t,T)P(t,T)}+\frac{1}{2}\mathrm{Var}_t^Q\left[\ln\left(S_T\right)\right]-\frac{\sigma_s^2}{2}\left(T-t\right)-\frac{1}{2}\int_t^T\beta_{v,T}^2dv}{\left(T-t\right)}.$$

B The log-likelihood function

In this section, we determine the log likelihood function (4).

Define $\mathbf{Y}_{t_i}=\left(\ln S_{t_i},\ln C_{t_i},\ln F\left(t_i,T_i,U_i\right),\ln F^*\left(t_i,T_i^*,U_i^*\right)\right)'$ and let $f_{\mathbf{Y}_{t_1},...\mathbf{Y}_{t_n}}\left(\bullet;\boldsymbol{\pi}_1,\boldsymbol{\pi}_2\right)$ denotes the joint density of the random vectors $\mathbf{Y}_{t_1},...\mathbf{Y}_{t_n}$ and $f_{\mathbf{Y}_{t_i}|\mathbf{Y}_{t_{i-1}}}\left(\bullet\left|\mathbf{y}_{t_{i-1}};\boldsymbol{\pi}_1,\boldsymbol{\pi}_2\right.\right)$ stands for the conditional density of $\mathbf{Y}_{t_i}$ given $\mathbf{Y}_{t_{i-1}}=\mathbf{y}_{t_{i-1}}$. The log-likelihood function associated with the observed sample $\mathbf{y}_{t_1},...,\mathbf{y}_{t_n}$ is

$$\begin{aligned}
\mathcal{L}\left(\boldsymbol{\pi}_1,\boldsymbol{\pi}_2;\mathbf{y}_{t_1},...\mathbf{y}_{t_n}\right) &= \ln f_{\mathbf{Y}_{t_1},...\mathbf{Y}_{t_n}}\left(\mathbf{y}_{t_1},...\mathbf{y}_{t_n};\boldsymbol{\pi}_1,\boldsymbol{\pi}_2\right) \\
&= \ln\prod_{i=1}^n f_{\mathbf{Y}_{t_i}|\mathbf{Y}_{t_{i-1}}}\left(\mathbf{y}_{t_i}\left|\mathbf{y}_{t_{i-1}};\boldsymbol{\pi}_1,\boldsymbol{\pi}_2\right.\right) \\
&= \sum_{i=1}^n\ln f_{\mathbf{Y}_{t_i}|\mathbf{Y}_{t_{i-1}}}\left(\mathbf{y}_{t_i}\left|\mathbf{y}_{t_{i-1}};\boldsymbol{\pi}_1,\boldsymbol{\pi}_2\right.\right) \\
&= -2n\ln\left(2\pi\right)-\frac{1}{2}\sum_{i=1}^n\ln\left|\boldsymbol{\Sigma}_{t_i}\right|-\frac{1}{2}\sum_{i=1}^n\left(\mathbf{x}_{t_i}-\boldsymbol{\mu}_{t_i}\right)'\boldsymbol{\Sigma}_{t_i}^{-1}\left(\mathbf{x}_{t_i}-\boldsymbol{\mu}_{t_i}\right).
\end{aligned}$$

According to Equations (3),

$$\boldsymbol{\mu}_{t_i} = \begin{pmatrix} \left(\alpha_S - \frac{\sigma_S^2}{2} - \theta\right)(t_i - t_{i-1}) - \left(\delta_{t_{i-1}} - \theta\right)\frac{1-\exp(-\kappa(t_i - t_{i-1}))}{\kappa} \\ \left(\alpha_C - \frac{\sigma_C^2}{2}\right)(t_i - t_{i-1}) \\ -\frac{1}{2}\int_{t_{i-1}}^{t_i}\left(\beta_{u,T_i} - \beta_{u,U_i}\right)^2 du - \lambda\int_{t_{i-1}}^{t_i}\left(\int_{T_i}^{U_i}\eta_{v,u}du\right)dv \\ -\frac{1}{2}\int_{t_{i-1}}^{t_i}\left(\beta^*_{u,U_i^*} - \beta^*_{u,T_i^*}\right)^2 du + \sigma_c\rho_{2,4}\int_{t_{i-1}}^{t_i}\left(\int_{T_i^*}^{U_i^*}\eta^*_{v,u}du\right)dv - \lambda^*\int_{t_{i-1}}^{t_i}\left(\int_{T_i^*}^{U_i^*}\eta^*_{v,u}du\right)dv \end{pmatrix}$$

and $\boldsymbol{\Sigma}_{t_i} = (\sigma_{t_i,\ell,j})_{\ell,j=1,2,3,4}$ where

$$\begin{aligned}
\sigma_{t_i,1,1} &= \left(\sigma_S - \frac{\sigma_\delta}{\kappa}\right)^2(t_i - t_{i-1}) + 2\left(\sigma_S - \frac{\sigma_\delta}{\kappa}\right)\frac{\sigma_\delta}{\kappa}\frac{1-\exp(-\kappa(t_i - t_{i-1}))}{\kappa} + \frac{\sigma_\delta^2}{\kappa^2}\frac{1-\exp(-2\kappa(t_i - t_{i-1}))}{2\kappa} \\
\sigma_{t_i,1,2} &= \rho_{12}\sigma_C\left(\left(\sigma_S - \frac{\sigma_\delta}{\kappa}\right)(t_i - t_{i-1}) + \frac{\sigma_\delta}{\kappa}\frac{1-\exp(-\kappa(t_i - t_{i-1}))}{\kappa}\right) \\
\sigma_{t_i,1,3} &= -\rho_{13}\int_{t_{i-1}}^{t_i}\left(\sigma_S - \sigma_\delta\frac{1-\exp(-\kappa(t_i - v))}{\kappa}\right)\left(\int_{T_i}^{U_i}\eta_{v,u}du\right)dv \\
\sigma_{t_i,1,4} &= -\rho_{14}\int_{t_{i-1}}^{t_i}\left(\sigma_S - \sigma_\delta\frac{1-\exp(-\kappa(t_i - v))}{\kappa}\right)\left(\int_{T_i^*}^{U_i^*}\eta^*_{v,u}du\right)dv \\
\sigma_{t_i,2,2} &= \sigma_C^2(t_i - t_{i-1}) \\
\sigma_{t_i,2,3} &= -\rho_{2,3}\sigma_C\int_{t_{i-1}}^{t_i}\left(\int_{T_i}^{U_i}\eta_{v,u}du\right)dv \\
\sigma_{t_i,2,4} &= -\rho_{2,4}\sigma_C\int_{t_{i-1}}^{t_i}\left(\int_{T_i^*}^{U_i^*}\eta^*_{v,u}du\right)dv \\
\sigma_{t_i,3,3} &= \int_{t_{i-1}}^{t_i}\left(\int_{T_i}^{U_i}\eta_{v,u}du\right)^2 dv \\
\sigma_{t_i,3,4} &= \rho_{3,4}\int_{t_{i-1}}^{t_i}\left(\int_{T_i}^{U_i}\eta_{v,u}du\right)\left(\int_{T_i^*}^{U_i^*}\eta^*_{v,u}du\right)dv \\
\sigma_{t_i,4,4} &= \int_{t_{i-1}}^{t_i}\left(\int_{T_i^*}^{U_i^*}\eta^*_{v,u}du\right)^2 dv.
\end{aligned}$$